Last class: 1-dim wave equation

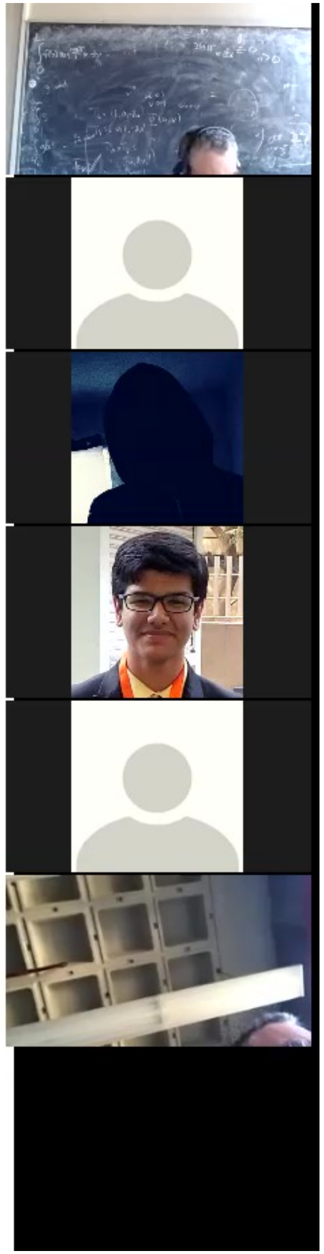
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{PDE}$$

where $c^2 = \frac{T}{S}$

$$u(0,t) = 0 = u(L,t) \qquad \text{BC}$$

$$u(x,0) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$



General Solution:

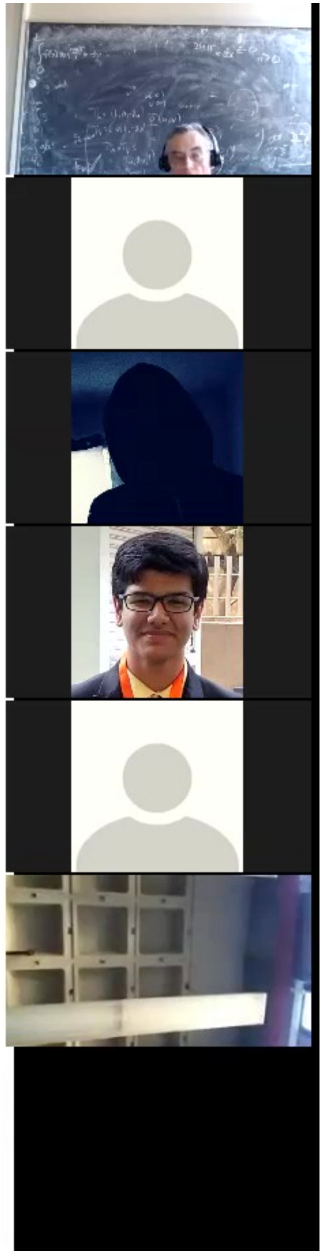
$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x \cos\frac{n\pi c}{L}t + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x \sin\frac{n\pi c}{L}t}$$

A_n's and B_n's can be calculated from IC

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x$$

$$g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \left(\sin\frac{n\pi}{L}x\right)\frac{n\pi c}{L}$$

$$\Rightarrow \boxed{A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x} \qquad \boxed{B_n\frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x}$$



Interpretation of Result

can rewrite

$$A_n \cos\frac{n\pi c}{L}x + B_n \sin\frac{n\pi c}{L}x =$$

as

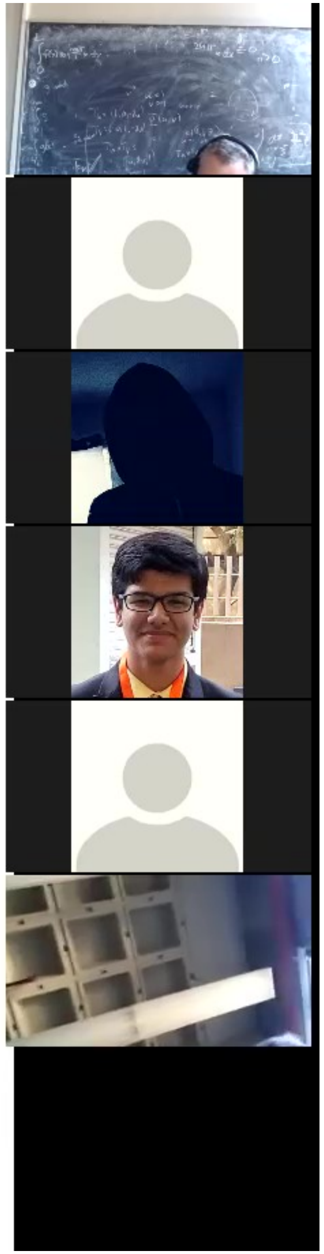
$$C_n \sin\left(\frac{n\pi c}{L}x + \theta_n\right)$$

where $C_n = \sqrt{A_n^2 + B_n^2}$

$$\theta_n = \tan^{-1}\left(\frac{A_n}{B_n}\right)$$

(idea: & we have $\left(\frac{A_n}{C_n}\right)^2 + \left(\frac{B_n}{C_n}\right)^2 = 1$

$\rightarrow$ determines angle θ_n s.t. $\sin\theta_n = \frac{A_n}{C_n}$, $\cos\theta_n = \frac{B_n}{C_n}$



use trig. identity

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

C_n called amplitude

the bigger C_n $\rightarrow$ the louder the sound

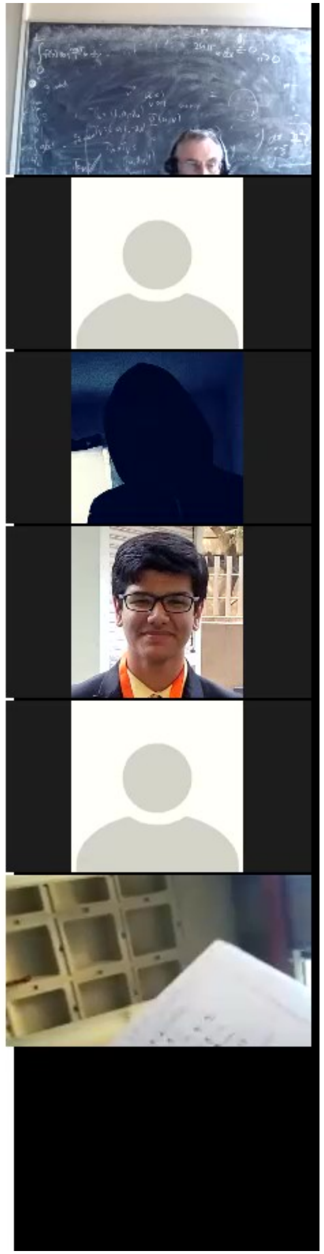
for example of guitar string

$\frac{n\pi c}{L}$ = frequency

the higher frequency $\rightarrow$ the higher the pitch

$c = \sqrt{\frac{T}{S}}$ to get higher pitch

- either tighten the tension
- use shorter string



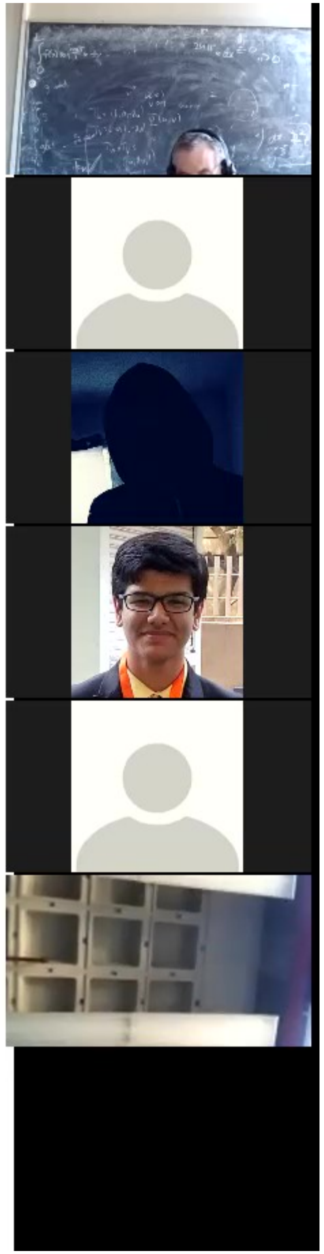
Higher dimensional wave equation

prime example: membrane with fixed boundaries (e.g. part of a drum)

$\Rightarrow$ get 2-dimensional wave equation

$$\boxed{\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\\ &= c^2\, \nabla^2 u\end{aligned}}$$

$u(x,y,t)$ = movement of part of membrane at (x,y) in z-direction at time t



use same strategy as before

$$u(x,t) = h(t)\ \phi(x,y)$$

separate time and space variables

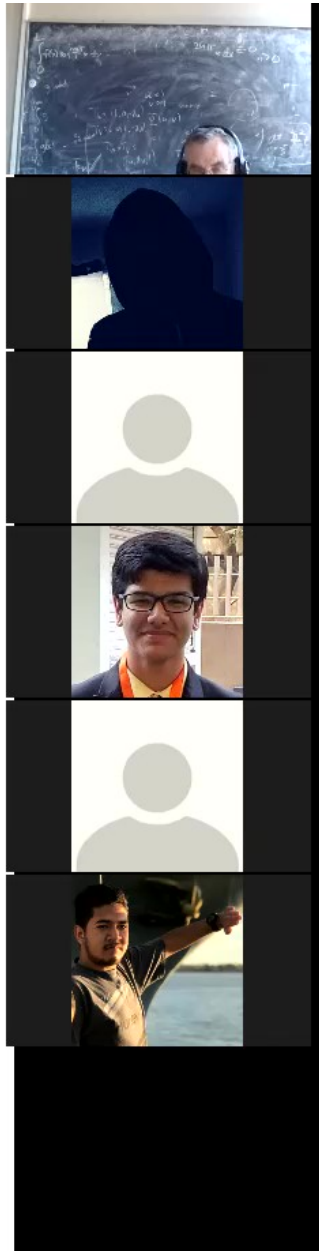
plug into PDE

$$h''(t)\,\phi(x,y) = c^2\left(h(t)\,\frac{\partial^2\phi}{\partial x^2}(x,y) + h(t)\,\frac{\partial^2\phi}{\partial y^2}\right)$$

$\frac{1}{c^2 h(t)\phi(x,y)}$

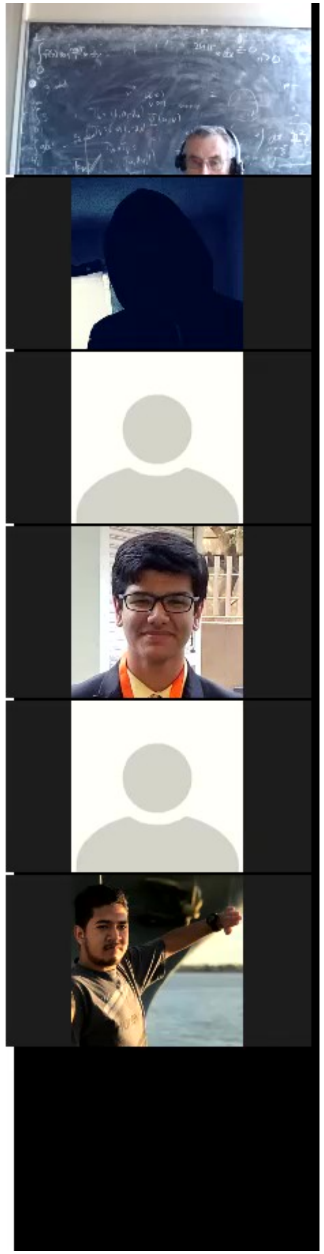
$$\frac{h''(t)}{c^2 h(t)} = \frac{\frac{\partial^2\phi(x,y)}{\partial x^2} + \frac{\partial^2\phi(x,y)}{\partial y^2}}{\phi(x,y)} = -\lambda$$

get ODE for $h(t)$
and PDE for $\phi(x,y)$

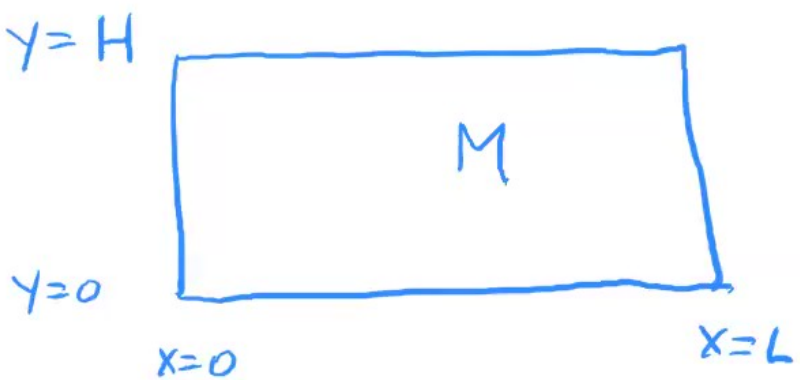


$$\boxed{\begin{gathered} h''(t) = -\lambda c^2 h(t) \\ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial^2 y} = -\lambda \phi \end{gathered}}$$

- want to use boundary conditions to calculate eigenvalues λ and eigenfunctions
- depends very much on shape of membrane!



7.3. Wave Equation for Rectangular Membrane

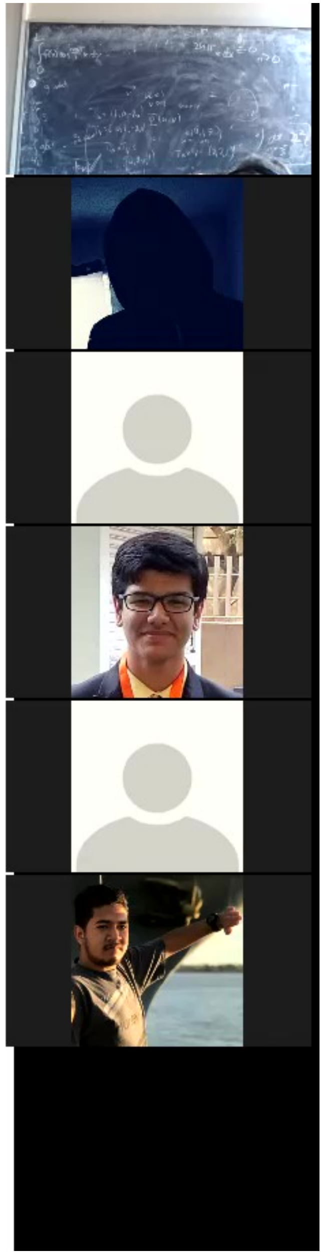


∂M = boundary of M
= 4 sides of rectangle

boundary conditions: $u|_{\partial M} = 0$ (short form)

explicitly:

$$\boxed{\begin{array}{llll} u(0,y,t)=0 & 0\le y\le H & & \\ u(L,y,t)=0 & 0\le y\le H, & t\ge 0 & \Big\}\ \text{left and right side} \\ u(x,0,t)=0 & 0\le x\le L & & \\ u(x,H,t)=0 & \text{"} & , t\ge 0 & \Big\}\ \text{bottom, top} \end{array}}$$



Translate into conditions for ϕ

$$\boxed{\begin{array}{ll} \phi(0,y) = 0 = \phi(L,y) & 0 \leq y \leq H \\ \phi(x,0) = 0 = \phi(x,H) & 0 \leq x \leq L \end{array}}$$

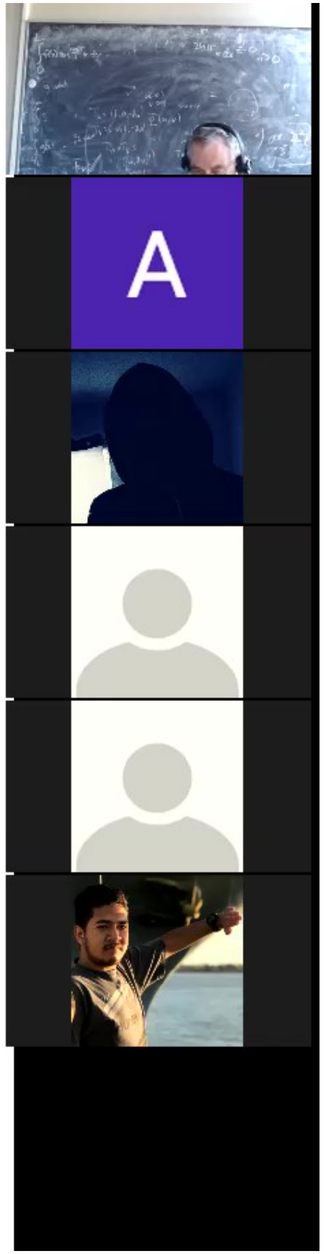
here: can further write ϕ as a product of 2 functions

$$\phi(x,y) = f(x)\, g(y)$$

plug into PDE for ϕ: $\nabla^2 \phi = -\lambda \phi$

$$\Rightarrow \quad f''(x) g(y) + f(x) g''(y) = -\lambda f(x) g(y) \qquad \Big| \; \frac{1}{f(x)g(y)}$$

$$\Rightarrow \quad \boxed{\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} = -\lambda}$$



Try to separate variables:

$$\frac{f''(x)}{f(x)} = -\lambda - \frac{g''(y)}{g(y)} = -\mu$$

$\Rightarrow$ get

$$\boxed{\frac{f''(x)}{f(x)} = -\mu}$$

and

$$\frac{g''(y)}{g(y)} = -\lambda - \underbrace{\frac{f''(x)}{f(x)}}_{=-\mu}$$

$\Rightarrow$

$$\boxed{\frac{g''(y)}{g(y)} = -(\lambda-\mu)}$$